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6 SEM TDC MTMH (CBCS) C 14

2022

(June/July)

MATHEMATICS

(Core)

Paper : C–14

(Ring Theory and Linear Algebra–II)

Full Marks : 80

Pass Marks : 32

Time : 3 hours

The figures in the margin indicate full marks for the questions

1. Answer any *three* from the following : 5×3=15

(a) State and prove division algorithm for $F[x]$, where F is a field. 5

(b) Define principal ideal domain (PID). If F is a field, then show that $F[x]$ is a principal ideal domain. 1+4

(c) Define irreducible polynomial and write an example. Let F be a field. If $f(x) \in F(x)$ and $\deg f(x) = 2$ or 3, then show that $f(x)$ is reducible over F if and only if $f(x)$ has a zero in F. 2+3

(d) In $Z[\sqrt{-5}]$, prove that $1+3\sqrt{-5}$ is irreducible but not prime. 5

2. Answer any *three* from the following : 5×3=15

(a) State and prove Eisenstein's criterion. 5

(b) Prove that a polynomial of degree n over a field has atmost n zeros counting multiplicity. 5

(c) Define unique factorization domain (UFD). Show that the ring $Z[\sqrt{-5}] = \{a+b\sqrt{-5} \mid a, b \in Z\}$ is an integral domain but not unique factorization domain. 1+4

(d) Define Euclidean domain. Prove that every Euclidean domain is a principal ideal domain. 1+4

3. Answer any *three* from the following : 6×3=18

(a) Suppose that V is a finite dimensional vector space with ordered basis $\beta = \{x_1, x_2, \cdots, x_n\}$. Let $f_i (1 \le i \le n)$ be the ith co-ordinate function with respect to β be defined such that $f_i(x_j) = \delta_{ij}$, where δ_{ij} is the Kronecker delta. Let $\beta^* = \{f_1, f_2, \cdots, f_n\}$. Then prove that β^* is an ordered basis for V^*, and for any $f \in V^*$, we have $f = \sum_{i=1}^{n} f(x_i) f_i$.

(b) For

$$A = \begin{bmatrix} 5 & -6 & -6 \\ -1 & 4 & 2 \\ 3 & -6 & -4 \end{bmatrix} \in M_{3\times 3}(R)$$

determine the eigenvalues of A and eigenspace of one eigenvalue of A.

(c) Let T be a linear operator on a finite dimensional vector space V. If f is the characteristic polynomial for T, then prove that the minimal polynomial divides the characteristic polynomial f for T.

(d) Find the characteristic polynomial and minimal polynomial for the real matrix

$$A = \begin{bmatrix} 7 & 4 & -1 \\ 4 & 7 & -1 \\ -4 & -4 & 4 \end{bmatrix}$$

Also show that the minimal polynomial divides the characteristic polynomial of A.

4. (a) Define invariant subspace of a vector space. 1

(b) Let T be a linear operator on R^3 such that $T(a, b, c) = (a+b+c, a+b+c, a+b+c)$. Let $W = \{(t, t, t) \mid t \in R\}$ be a subspace of R^3.

Show that—

(i) W is a T-invariant subspace of R^3

(ii) the characteristic polynomial of T_W divides the characteristic polynomial of T. 6

Or

Let V be a finite-dimensional vector space over the field F and let T be a linear operator on V. Then show that T is diagonalizable if and only if the minimal polynomial for T has the form

$$p = (x - c_1) \cdots (x - c_k)$$

where $c_1, c_2, \cdots, c_k$ are distinct elements of F.

5. *(a)* If V is an inner product space, then for any vectors α, β in V and any scalar c, prove that $\|\alpha + \beta\| \leq \|\alpha\| + \|\beta\|$. 5

Or

Let V be an inner product space and let $\beta_1, \beta_2, \cdots, \beta_n$ be any independent vectors in V. Then construct orthogonal vectors $\alpha_1, \alpha_2, \cdots \alpha_n$ in V such that for each $k = 1, 2, \cdots, n$, the set $\{\alpha_1, \alpha_2, \cdots, \alpha_k\}$ is a basis for the subspace spanned by $\beta_1, \beta_2, \cdots, \beta_k$.

(b) Define orthogonal vectors. Consider the vectors $\beta_1 = (3, 0, 4)$, $\beta_2 = (-1, 0, 7)$, $\beta_3 = (2, 9, 11)$ in R^3 equipped with standard inner product. Apply the Gram-Schmidt orthogonalisation process to find orthogonal vectors corresponding to the given vectors. 1+4

(c) For any linear operator T on a finite dimensional inner product space V, prove that there exists a unique linear operator T^* on V such that $(T\alpha|\beta) = (\alpha|T^*\beta)$ for all $\alpha, \beta \in V$. 5

6. (a) Define adjoint of a linear operator T on a vector space V. Give an example of adjoint of a linear operator T on V. 2

(b) Answer any *two* questions from the following : 4×2=8

(i) Let V be a finite-dimensional inner product space. If T and U are linear operator on V, then prove that

(1) $(T+U)^* = T^* + U^*$

(2) $(T^*)^* = T$

(ii) Let $\{\alpha_1, \cdots, \alpha_n\}$ be an orthogonal set of non-zero vectors in an inner product space V. If β is any vector in V, then prove that

$$\sum_k \frac{|(\beta|\alpha_k)|^2}{||\alpha_k||^2} \leq ||\beta||^2$$

(iii) Let V be a finite-dimensional inner product space, and f be a linear functional on V. Then show that there exists a unique vector β in V such that $f(\alpha) = (\alpha|\beta)$ for all α in V.

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