6 SEM TDC MTMH (CBCS) C 14

2025

(May)

MATHEMATICS

(Core)

Paper: C-14

(Ring Theory and Linear Algebra—II)

Full Marks: 80

Pass Marks: 32

Time: 3 hours

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The figures in the margin indicate full marks for the questions

- 1. (a) If F is commutative, then write the condition such that F[x] is invertible.
 - (b) Prove that every Euclidean domain possesses unity.
 - (c) Show that $x^2 + 3x + 2$ has four zeros in Z_6 .

(Turn Over)

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i,		1	
	(d)	Let F be a field. Then prove that the ring of polynomial $F[x]$ is principal ideal domain (PID).	4
	(e)	Prove that a polynomial of degree n over a field has at most n zeros, counting multiplicity.	6
		Or	
	2 , 2	Let F be a field and let $f(x)$, $g(x) \in F[x]$ with $g(x) \neq 0$. Then prove that there exist unique polynomials $q(x)$ and $r(x)$ in $F[x]$ such that $f(x) = g(x)q(x) + r(x)$	
		and either $r(x) = 0$ or $\deg r(x) < \deg g(x)$.	
2.	(a)	What is the inverse of $1+\sqrt{2}$ in $Z[\sqrt{2}]$?	1
	(b)	Define Euclidean domain.	1
	(c)	Test the irreducibility of the polynomial $x^5 + 9x^4 + 12x^2 + 6$ in Q .	2
		Prove that in a principal ideal domain,	
		an element is irreducible if and only if it is a prime.	5
		Define unique factorization domain and prove that every field is unique factorization domain. 1+5	5=6





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- Prove that $Z[\sqrt{3}] = \{a + b\sqrt{3} \mid a, b \in Z\}$ is a Euclidean domain.
- 3. (a) Write when two linear functionals are said to be equal on a vector space V(F).
 - (b) Define invariant subspace.
 - (c) If S_1 and S_2 are two subsets of a vector space V(F) such that $S_1 \subseteq S_2$, then show that $S_2 \subseteq S_1$.
 - (d) Prove that the subspace spanned by two subspaces each of which is invariant under some linear operator T, is itself invariant under T.
 - (e) Let V be an n-dimensional vector space over the field F and let

$$\beta = \{\alpha_1, \alpha_2, \cdots, \alpha_n\}$$

be a basis for V. Then prove that there is a uniquely determined basis

$$\beta' = \{f_1, f_2, \cdots, f_n\}$$

for V' such that $f_i(\alpha_j) = \delta_{ij}$.

(Turn Over)



Let V be finite dimensional vector space over the field F and let W be a subspace of V. Then prove that

 $\dim W + \dim W^{\circ} = \dim V$

- 4. (a) Write about the eigenvalues and eigenvectors of the identity matrix.
 - (b) If V is n-dimensional vector space, then what is the condition that the linear operator T is diagonalizable?
 - (c) Test the diagonalizability of the following matrix:

$$\begin{bmatrix} \frac{1}{2} & \frac{3}{2} \\ \frac{3}{2} & \frac{1}{2} \end{bmatrix}$$

(d) Define minimal polynomial and show that the minimal polynomial of the real matrix

네트를 가는 나는 아이들이 되었다. 함께를 되었다.

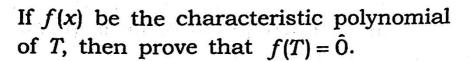
$$\begin{bmatrix} 5 & -6 & +6 \\ -1 & 4 & 2 \\ 3 & -6 & -4 \end{bmatrix}$$

is
$$(x-1)(x-2)$$
.

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Write the only vector that is orthogonal (a) to itself.

Define orthogonal complement. (b)

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If α , β are vectors in an inner product (c) space V, then prove that

$$\|\alpha + \beta\| \le \|\alpha\| + \|\beta\|$$

$$Or$$

If W_1 and W_2 are subspaces of a finite dimensional inner product space, then prove that

$$(W_1 + W_2)^{\perp} = W_1^{\perp} \cap W_2^{\perp}$$

 $\beta = \{\alpha_1, \alpha_2, \dots, \alpha_m\}$ is any finite (d)orthonormal set in an inner product space V, and if β is any vector in V, then prove that

$$\sum_{i=1}^{m} \left| (\beta, \alpha_i) \right|^2 \le \left\| \beta \right\|^2$$





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In an inner product space, prove that

$$|(\alpha, \beta)| \le ||\alpha|| ||\beta||$$

6. (a) Write the two self-adjoint operators on any inner product space V(F).

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(b) Define normal operator.

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(c) If T_1 and T_2 are normal operators on an inner product space with the property that either commutes with the adjoint of the other, then prove that T_1 T_2 is also normal operator.

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(d) Let V be the direct sum of its subspaces W_1 and W_2 . If E_1 is the projection on W_1 along W_2 , and E_2 is the projection on W_2 along W_1 , then prove that—

(i) $E_1 + E_2 = I$;

(ii)
$$E_1 E_2 = \hat{0}, E_2 E_1 = \hat{0}$$

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(e) If T_1 and T_2 are self-adjoint linear operators on an inner product space V, then prove that (i) $T_1 + T_2$ is self-adjoint and (ii) if $T_1 \neq \hat{0}$ and a is a non-zero scalar, then aT_1 is self-adjoint iff a is real.

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Or

Apply the Gram-Schmidt process to the vectors (1,0,0), (1,1,0), (1,1,1) to obtain an orthonormal basis for $V_3(R)$ with the standard inner product.

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