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**4 SEM TDC MTMH (CBCS) C 9**

**2023**

( May/June )

**MATHEMATICS**

( Core )

Paper : C-9

**( Riemann Integration and Series  
of Functions )**

*Full Marks : 80*

*Pass Marks : 32*

*Time : 3 hours*

*The figures in the margin indicate full marks  
for the questions*

1. (a) State the two vital requirements for  
existence of

$$\int_a^b f(x) dx$$

1+1=2

( 2 )

(b) Show that if  $f \in R[a, b]$ , then the value of  $\int_a^b f(x) dx$  is unique. 3

Or

Show that every constant function is integrable.

2. (a) Let  $P = \{([x_{i-1}, x_i]), t_i\}_{i=1}^n$  be a tagged partition of  $I = [a, b]$ . Then define Riemann sum of  $f : [a, b] \rightarrow \mathbb{R}$ . Give an example of the Riemann sum if  $I = [1, 2]$ . 2

(b) Let  $P = \{([x_{i-1}, x_i]), t_i\}_{i=1}^n$  be a tagged partition of  $I = [a, b]$ . Then show that  $S(kf, P) = kS(f, P)$ . 3

(c) Answer any four questions from the following : 5×4=20

(i) Write an example with explanation thereof that all bounded functions are not Riemann integrable.

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( Continued )

( 3 )

(ii) Let  $f : [a, b] \rightarrow \mathbb{R}$  is such that if  $x_1 < x_2$ , then  $f(x_1) \leq f(x_2)$ . Show that  $f \in R[a, b]$ .

(iii) Let  $f : [a, b] \rightarrow \mathbb{R}$  be integrable. Then  $|f|$  is integrable and show that

$$\left| \int_a^b f(x) dx \right| \leq \int_a^b |f(x)| dx$$

(iv) Let  $f, g : [a, b] \rightarrow \mathbb{R}$  be integrable and  $f(x) \leq g(x) \forall x \in [a, b]$ . Then show that

$$\int_a^b f(x) dx \leq \int_a^b g(x) dx$$

(v) Let  $f : [a, b] \rightarrow \mathbb{R}$  be integrable.

Define  $F$  on  $[a, b]$  as

$$F(x) = \int_a^x f(t) dt \quad \text{where } x \in [a, b].$$

Show that  $F$  is differentiable at  $c \in [a, b]$  and  $F'(c) = f(c)$ .

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( Turn Over )

( 4 )

3. (a) Show that

(i)  $\Gamma(1) = 1$

(ii)  $\Gamma(n+1) = n\Gamma(n)$  1+2=3

(b) Show that if  $m \in \mathbb{N}$ , then  $\Gamma(m+1) = \underline{m}$ . 3

(c) Discuss the convergence of beta function. 4

Or

Show that  $\int_{-\infty}^{\infty} e^{-u^2} du = \sqrt{\pi}$  and hence

show that  $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$ .

4. (a) State whether true or false : 1

Pointwise convergence implies uniform convergence.

(b) Let  $(f_n)$  be a real sequence of functions defined on a finite set  $X = \{a_1, \dots, a_k\}$  converging pointwise to a function  $f : X \rightarrow \mathbb{R}$ . Establish that the convergence is uniform. 2

( 5 )

(c) Let  $(f_n)$  be a sequence of integrable functions on  $[a, b]$ . Let  $f_n \rightarrow f$  uniformly on  $[a, b]$ . Show that  $f$  is integrable on  $[a, b]$  and

$$\int_a^b f(x) dx = \lim \int_a^b f_n(x) dx \quad 4$$

(d) Show that if  $(f_n)$  be a uniformly Cauchy sequence on a set  $X$  in  $\mathbb{R}$ , then it converges to  $f : X \rightarrow \mathbb{R}$  uniformly. 4

(e) Show that the series

$$\sum_{n=1}^{\infty} \frac{x}{(1+nx^2)^n}$$

converges uniformly on any interval  $[a, b]$ . 4

(f) State and prove Cauchy's criterion for the uniform convergence of a series. 5

( 6 )

(g) Let  $f_n : (a, b) \rightarrow \mathbb{R}$  be differentiable and the sequence  $(f'_n)$  converges uniformly to  $g : (a, b) \rightarrow \mathbb{R}$ . Let there exists  $c \in (a, b)$  such that the sequence  $(f_n(c))$  converges. Then show that the sequence  $(f_n)$  converges uniformly to a continuous function  $f : (a, b) \rightarrow \mathbb{R}$ . 5

5. (a) State whether true or false : 1

A power series is a particular case of infinite series of functions

$$\sum_{n=0}^{\infty} f_n(x)$$

(b) Let  $\sum_{n=0}^{\infty} a_n(x-a)^n$  be a power series.

Show that there exists a unique extended real number  $R$ ,  $0 \leq R < \infty$ , such that  $\forall x$  with  $|x-a| < R$ , the series converges absolutely and uniformly to a function  $f$  on  $(-r, r)$ ;  $0 < r < R$ . 4

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( Continued )

( 7 )

(c) Given a power series  $\sum_{n=0}^{\infty} a_n(x-a)^n$ , determine an extended real number  $R$

such that  $\frac{1}{R} = \limsup |a_n|^{1/n}$ . 5

(d) State and prove Abel's limit theorem. 5

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